SEVERAL NONLINEAR PROBLEMS IN TRANSIENT FILTRATION

(O NEKOTORYKH NELINEINYKH ZADACHAKH NESTATSIONARNOI FIL'TRATSII)

PMM Vol.26, No.1, 1962, pp. 196-201

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(Received October 13, 1961)

Problems of unsteady or transient filtration of a liquid or gas through a porous medium [1] reduce to a nonlinear differential equation as follows

$$\frac{\partial u}{\partial t} = \frac{\partial^2 \varphi(u)}{\partial x^2} \qquad (\varphi(u) \geqslant 0, \ \varphi'(u) \geqslant 0, \ u \geqslant 0) \tag{0.1}$$

In particular we have for the case of isothermal gas motion and motion of subterranean water the equation

$$\frac{\partial u}{\partial t} = a^2 \frac{\partial^2 u^2}{\partial x^2} \qquad \left(a = \frac{k}{2m\mu}\right) \tag{0.2}$$

In this expression $u(x, t)$ is the gas density of the subterranean water head, m is the porosity of the subsoil, k is the permeability of the porous terrain, and μ is the viscosity of the gas.

Boundary problems are of interest in which gas density or pressure is known at the boundary of the strata, or the head acting on the subterranean water. This leads us to boundary conditions of the following kind

$$u(0, t) = F(t)$$

If the flow of gas or underground water is given at the boundaries, the following boundary conditions accrue

$$\frac{\partial \varphi(0, t)}{\partial x} = F_1(t), \qquad \frac{\partial \varphi(l, t)}{\partial x} = F_2(t)$$

According to the sense of the problem $F(t) \geqslant 0$, $F_1(t) \leqslant 0$, $F_2(t) \geqslant 0$. The derivative $\partial \phi(u)/\partial x$ (gas flow) is a continuous function. Equations (0.1) and (0.2) are dealt with in [2-4]. The problems of numerical solution of equations of the type (0.1), (0.2) where the initial and the boundary problems are strictly positive are dealt with in [5-7]. In our present work results are given of calculations of several actual problems on the BESM-2 computer at the Computing Centre of the Academy of Sciences.

1. The order of Equation (0.1) depends on the value of the function $u(x, t)$; when $u > 0$ it is a second order parabolic equation, when $u = 0$ it degenerates into a first order equation. Self-similar solutions of Equation (0.1) are constructed in [2], and these have a break at the abscissae depending on time, at which $\partial u/\partial x$ undergoes a finite or infinite jump. For this reason the function $u(x, t)$ will not evince the smoothness prescribed by the equation at these points, and in fact, it will be a generalized solution. It is indeed the break point (discontinuity) which gives rise to the main difficulties of numerical solution and is the deciding factor as regards choice of method. Difference methods, built up without regard to this peculiarity of the solution, can, in some cases, give a qualitatively incorrect result. The existence and uniqueness of a solution of (0.1) for the case of degeneration are dealt with in [3]. In [8-10] problems concerning the fundamentals and methods of numerical calculation of a general solution to (0.1) are studied. The Cauchy problem is studied, and also the first and second boundary problems for $0 < x < \infty$ and $0 < x < l$. Analysis demonstrates that for numerical calculation of (0.1) it is convenient to adopt the "explicit" scheme, i.e. to replace (0.1) by the following difference analogue

$$u_{ik+1} = u_{ik} + \frac{\tau}{h^2}\left[\varphi(u_{i+1k}) - 2\varphi(u_{ik}) + \varphi(u_{i-1k})\right] \qquad (1.1)$$

where h, τ are respectively the pitches (or steps) in the spatial and the time coordinates. The approximate solution obtained from (1.1) shares the main features of the accurate one; it is non-negative, it is limited (or bounded) (that is it does not exceed the maximum value of the initial and the boundary function); it approaches the accurate solution as the step is indefinitely decreased.

2. The problem now dealt with is that of the filtration of a semi infinite stratum: $0 < x < \infty$. The distribution of head satisfies Equation (0.2). The initial and boundary problems are as follows

$$u|_{t=0} \equiv 0, \quad u|_{x=0} = at - bt^2 \qquad (a > 0,\ b > 0) \qquad (2.1)$$

The boundary condition corresponds to the head at the boundary which varies nonmonotonically; at first the head increases, then it decays. Thus the liquid first of all penetrates the stratum, and then it begins to flow out of it. It is of interest to determine the instant of time t_0

when liquid begins to flow out of the stratum, i.e. when the derivative $\partial^2 u/\partial x^2$ vanishes when $x = 0$. The calculation was done by the difference method (1.1). Figures 1 and 2 give graphical solutions with boundary conditions of the type (2.1) with $a = 1/2$, $b = 1$ and $a = 1/2$, $b = 1/4$ for

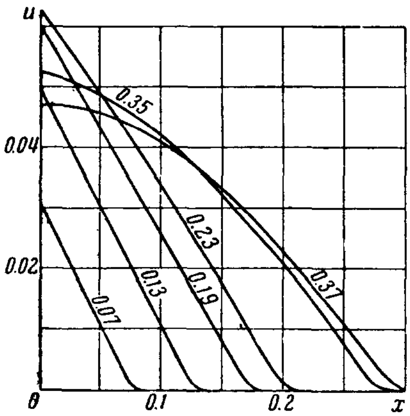

Fig. 1.

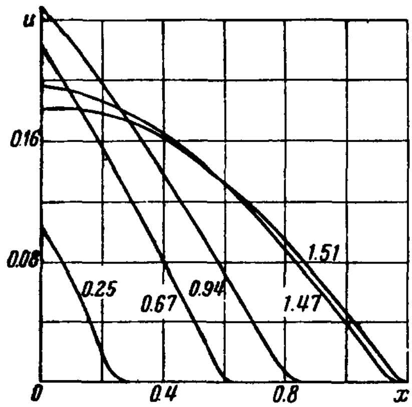

Fig. 2.

various times t (instant t is indicated at the side of the curve to which it applies). The table gives values of the solution to problem (0.2) to (2.1) for $a = 1/2$, $b = 1/4$. The graphs show that instant t_0 equals 0.37 and 1.47 for the cases $a = 1/2$, $b = 1$ and $a = 1/2$, $b = 1/4$ respectively. Instant t_1, when u is a maximum for $x = 0$ equals 0.25 and 1.00 respectively. It is evident that in both cases $t_0 > t_1$.

3. In the isothermal gas filtration problem in a semi-infinite stratum ($0 < x < \infty$), when the gas pressure at the boundary is such that when $t \to \infty$ the solution attains a self-similar regime, it is interesting to try and calculate so as to analyse the velocity with which the self-similarity regime is attained.

Suppose the initial and boundary functions are as follows

$$u|_{t=0} \equiv 0, \qquad u|_{x=0} = \sigma t^p + \sigma_1(t) \geqslant 0 \tag{3.1}$$

In this expression, $\sigma_1(t)$ is such that

$$\lim \sigma_1(t) / t^p = 0 \quad \text{when } t \to \infty$$

It has been shown in [2] that when $\sigma_1(t) \equiv 0$ the solution to problem (0.2) to (3.1) is self similar, i.e.

$$u(x,t) = \sigma t^p f(\xi), \qquad \xi = \frac{x}{\sqrt{\sigma t^{p+1}}}$$

TABLE OF VALUES OF $u(x,\ t) \times 10^4$

x \ t	0.077	0.157	0.250	0.447	0.668	0.936	1.19	1.41	1.46	1.48	1.53
0.000	370	724	1094	1736	2225	2490	2410	2073	1958	1904	1803
0.025	248	608	983	1637	2142	2427	2372	2064	1958	1908	1814
0.050	126	490	871	1537	2057	2364	2333	2053	1955	1908	1821
0.075	23	373	759	1437	1971	2299	2292	2039	1948	1905	1824
0.100		254	646	1336	1885	2233	2249	2022	1938	1889	1824
0.125		136	533	1234	1798	2166	2205	2003	1926	1889	1820
0.150		29	419	1132	1710	2098	2160	1981	1911	1877	1813
0.175		1	305	1029	1622	2029	2113	1957	1893	1862	1803
0.200			190	926	1532	1956	2064	1932	1873	1845	1790
0.225			73	822	1442	1888	2014	1904	1851	1825	1775
0.250			6	717	1352	1816	1963	1874	1826	1803	1758
0.275				612	1260	1744	1911	1842	1800	1779	1738
0.300				506	1168	1670	1857	1808	1771	1752	1715
0.325				400	1075	1595	1802	1773	1741	1724	1691
0.350				203	982	1520	1746	1736	1709	1694	1665
0.375				186	888	1443	1668	1697	1675	1662	1637
0.400				77	793	1366	1630	1657	1639	1629	1607
0.425				7	697	1288	1570	1615	1602	1593	1575
0.450					601	1209	1509	1572	1563	1556	1541
0.475					505	1129	1447	1527	1522	1518	1506
0.500					407	1049	1384	1481	1481	1478	1469
0.525					309	967	1320	1434	1437	1436	1431
0.550					210	885	1255	1386	1393	1393	1391
0.575					111	802	1189	1336	1347	1349	1349
0.600					22	719	1122	1285	1299	1303	1307
0.625						634	1054	1232	1251	1256	1263
0.650						549	986	1179	1201	1208	1217
0.675						463	916	1124	1150	1159	1171
0.700						377	845	1068	1098	1108	1123
0.725						289	774	1012	1044	1056	1074
0.750						201	701	954	990	1003	1023
0.775						112	628	895	934	949	972
0.800						28	554	835	878	894	919
0.825						1	479	774	820	838	866
0.850							403	712	761	780	811
0.875							326	649	702	722	755
0.900							249	585	641	663	698
0.925							171	520	579	602	640
0.950							91	454	517	541	581
0.975							20	388	453	479	521
1.000							0	320	389	416	460
1.025								252	323	352	399
1.050								182	257	287	336
1.075								112	190	221	272
1.100								41	122	154	208
1.125								3	52	86	142
1.150									5	22	76
1.175											16
1.200											0

A point ξ_0 exists such that

$$f(\xi)>0 \quad \text{when } \xi<\xi_0, \qquad f(\xi)\equiv 0 \quad \text{when } \xi \geqslant \xi_0$$

to which the following apply

$$u(x,t)>0 \quad \text{when } x>\xi_0 \sqrt{\sigma t^{p+1}}, \qquad u\equiv 0 \quad \text{when } x \geqslant \xi_0 \sqrt{\sigma t^{p+1}}$$

With the elapse of time the point $x_0 = \xi_0 \sqrt{\sigma t^{p+1}}$ moves to the right along the abscissa and it becomes rather difficult to work out a practical solution for fairly high values of t close to $x_0(t)$. Because the solution to problem (0.2) to (3.1) attains the self-similar regime when $t \to \infty$ [9], i.e.

$$\lim \frac{u(x,t)}{t^p} = \sigma f(\xi) \quad \text{when } t\to\infty$$

there is good reason to go over to moving "self-similar" coordinates, in which the solution at fairly high values of t hardly varies. Thus the following transformation of variables is convenient

$$\rho = \frac{u}{(t+1)^p}, \qquad \xi = \frac{x}{\sqrt{\sigma(t+1)^p}}, \qquad \eta = \ln(t+1) \tag{3.2}$$

With this transformation we have

$$\lim \rho(\xi,\eta) = f(\xi) \quad \text{when } \eta\to\infty$$

and this is very convenient for practical calculation at high values of t. It is evident from Formula (3.2) that $t = e^{\eta} - 1$, i.e. for comparatively small values of η the time t is already great. (The shift along the t axis in the Formulas (3.2) is carried out for convenience of calculation close to the point $t = 0$). On changing variables (3.2) problem (0.2) to (3.1) transforms into the boundary problem

$$\frac{\partial\rho}{\partial\eta} = \frac{\partial^2\rho^2}{\partial\xi^2} + \frac{p+1}{2}\xi\frac{\partial\rho}{\partial\xi} - p\rho \qquad (0\leqslant\xi<\infty,\ 0\leqslant\eta<\infty,\ \rho\geqslant 0) \tag{3.3}$$

$$\rho\equiv 0 \quad \text{when } \eta=0, \qquad \rho = \sigma(1-e^{-\eta})^p + \sigma_1(t(\eta))\,e^{-\eta p} \quad \text{when } \xi = 0 \tag{3.4}$$

Here the derivative $\partial\rho/\partial\xi$ is discontinuous, and this involves further difficulties for approximating it by finite difference. In the first place the error of the approximation of replacing $\partial\rho/\partial\xi$ by a finite difference at a point where $\partial\rho/\partial\xi$ has a break does not tend to zero with indefinite decrease in the step. Further analysis is essential (for instance consult [9]) for a proof of convergence between a difference solution and the accurate one. Furthermore it has been shown by calculation that the derivative $\partial \rho / \partial \xi$ cannot be approximated, for instance by a central difference for then the explicit scheme becomes unstable (starting at fairly low values of ρ_{ik}, i.e. close to the discontinuity point ξ_0, the graph of the solution oscillates about the axis and rapidly goes out of hand). Further, the implicit scheme is stable, but the approximate solution obtained oscillates about the abscissa and takes on negative values which has a qualitative effect on the solution.

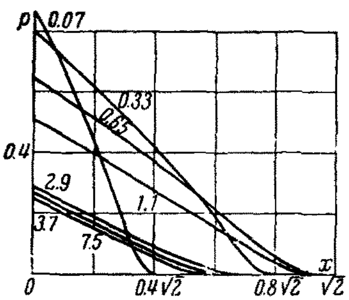

Fig. 3.

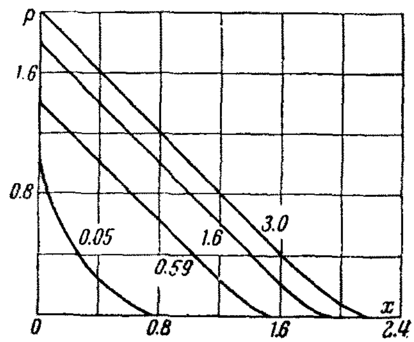

Fig. 4.

Two methods of approximating $\partial \rho / \partial \xi$ are proposed in [9].

The first method consists in replacing $\partial \rho / \partial \xi$ by a "right-hand side difference" so that the difference solution ρ_{ik} is determined from the formula

$$\rho_{ik+1} = \rho_{ik}(1 - p\tau) + i\tau(\rho_{i+1k} - \rho_{ik})\frac{p+1}{2} + \frac{\tau}{h^2}(\rho^2_{i+1k} - 2\rho^2_{ik} + \rho^2_{i-1k})$$

$$\rho_{i0} \equiv 0, \qquad \rho_{0k} = \sigma(1 - e^{-k\tau})^p + \sigma_1(t(k\tau))\, e^{-kp\tau} \tag{3.5}$$

Solution (3.5) is stable, it approaches (converges with) the accurate one when $h \to 0$ if the step in time τ satisfies the condition that $\tau \leqslant Ah^2$, where A is a definite (determined) constant.

The second way of replacing (3.3) by a difference equation consists in bringing (3.3) into the form

$$\frac{\partial \rho}{\partial L} = \frac{\partial^2 \rho^2}{\partial \xi^2} - p\rho \qquad \left(\frac{\partial \rho}{\partial L} = \frac{\partial \rho}{\partial t} - \frac{p+1}{2}\xi\frac{\partial \rho}{\partial \xi}\right) \tag{3.6}$$

and the expression $\partial \rho / \partial L$ is approximated by an oblique difference.

Equation (3.6) can be solved in a manner analogous to (1.1) with the difference only that the correspondence between the points of the solution for $\eta = \eta_1$ and $\eta = \eta_1 + r$ is taken along direction L, depending on ξ

$$\tan(L, \xi) = \frac{-2}{(p+1)\xi}$$

Such a "slanted" arrangement of points of the solution for $\eta = \eta_1$ and $\eta = \eta_1 + r$ corresponds to a straight line in the first system of coordinates. When $\xi = 0$ the angle (L, ξ) is equal to $\pi/2$ so that the boundary condition is given accurately. The difference scheme is as follows

$$\rho_{ik+1} = [\rho_{i+\gamma+1,k}\,\alpha + \rho_{i+\gamma,k}\,(1-\alpha)]\,(1-p\tau) + \tag{3.7}$$

$$+ \frac{\tau}{h^2}\,[\alpha(\rho^2_{i+\gamma+2,k} - 2\rho^2_{i+\gamma+1,k} + \rho^2_{i+\gamma,k}) + (1-\alpha)(\rho^2_{i+\gamma+1,k} - 2\rho^2_{i+\gamma,k} + \rho^2_{i+\gamma-1,k})]$$

$$\gamma = \left[\frac{p+1}{2}\, i\tau\right], \qquad \alpha = \left\{\frac{p+1}{2}\, i\tau\right\}$$

The difference solution obtained from (3.5) and (3.7) is bounded, has a finite number of regions where it is monotonic for fixed value of k and embodies a finite velocity of disturbance propagation.

It is demonstrated in [9] that the order of error in (3.5) is $O(\sqrt{h})$. It is easy to see that the order of error of (3.7) is less because in this case the term $\partial\rho/\partial\xi$ is absent. Figures 3 and 4 show graphs of calculations of problem (0.2) to (3.1) for $\sigma_1 = 1$, $\sigma = 1/4$, 2, $p = 1$. It is evident that a rapid change takes place from one self-similar solution to the other.

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Translated by V.H.B.